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Analytic Index and Chiral Fermions

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Abstract: A recent application of an index relation of the form, $dim\ ker\ M - dim\ ker\ M^{\dagger} = \nu$, to the generation of chiral fermions in a vector-like gauge theory is reviewed. In this scheme the chiral structure arises from a mass term with a non-trivial index.The essence of the generalized Pauli-Villars regularization of chiral gauge theory,which is based on this mechanism,is also clarified.

Keywords: chiral fermions, vector-like gauge theory, analytic index

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I. Introduction

The notion of index plays an important role in quantum field theory. The best known example may be the Atiyah-Singer index theorem [1] and chiral anomaly. In a recent article, Jackiw [2] accounted his encounter with the notion of index in the study of chiral anomaly and emphasized the importance of various indices . The Riemann-Roch theorem ,which may be regarded as a part of the Atiyah-Singer index theorem ,appears as a ghost number anomaly [3] in two-dimensional quantum gravity. The Witten index in supersymmetric theory [4] is also related to some of topological anomalies [5].

The index associated with a linear operator M is written as

$$dim\ ker\ M - dim\ ker\ M^\dagger = \nu \tag{1}$$

where ν stands for an integer and it is called an index. The above form of index is also called an analytic index. In eq.(1) *dim ker M* stands for the number of normalizable solutions u_n of

$$Mu_n = 0 \tag{2}$$

The index relation (1) is also written as

$$dim\ ker\ M^\dagger M - dim\ ker\ MM^\dagger = \nu \tag{3}$$

The equivalence of these two specifications is seen by noting that $Mu = 0$ implies $M^\dagger Mu = 0$. Conversely,$M^\dagger Mu = 0$ implies $(M^\dagger Mu, u) = (Mu, Mu) = 0$ and thus $Mu = 0$ if the inner product is positive definite.

The index is an integer and as such it is expected to be invariant under a wide class of continuous deformation of parameters characterizing the operator M.

In the present article, I would like to review a recent application of the notion of index to the generation of chiral fermions via a mass matrix with a non-trivial index. The basic mechanism of the generalized Pauli-Villars regularization of chiral gauge theories, which is based on this scheme, is also clarified.

II. Chiral fermions in a vector-like scheme and a mass matrix with non-trivial index

The fundamental fermions appearing in the unified theory of electroweak interactions have a chiral structure. At this moment, it is not known how this chiral structure arises ; it might be that the basic structure of nature has a chiral structure. On the other hand, in some models of fundamental fermions such as a vector-like scheme, one envisions the appearance of the chiral structure as a result of some dynamical effects. Although no definite dynamical mechanism which realizes this idea is known, the recent suggestion by Narayanan and Neuberger[6]on the basis of an analytic index gives an interesting and suggestive kinematical picture. To be specific, their idea is to start with a vector-like Lagrangian for an $SU(2)\times U(1)$ gauge theory, for example, written in an abbreviated notation

$$\mathcal{L}_L = \overline{\psi} i\gamma^\mu D_\mu \psi - \overline{\psi}_R M \psi_L - \overline{\psi}_L M^\dagger \psi_R \tag{4}$$

with

$$\not{D} = \gamma^\mu(\partial_\mu - igT^a W_\mu^a - i(1/2)g' Y_L B_\mu) \tag{5}$$

and $Y_L = 1/3$ for quarks and $Y_L = -1$ for leptons. The field ψ in (4) is a column vector consisting of an infinite number of $SU(2)$ doublets, and the infinite dimensional *nonhermitian* mass matrix M satisfies the index condition

$$\dim\ker(M^\dagger M) - dim\ker(MM^\dagger) = 3 \tag{6}$$

and $dim\ker(MM^\dagger) = 0$.
In the explicit "diagonalized" expression of M

$$M = \begin{pmatrix} 0 & 0 & 0 & m_1 & 0 & 0 & .. \\ 0 & 0 & 0 & 0 & m_2 & 0 & .. \\ 0 & 0 & 0 & 0 & 0 & m_3 & .. \\ . & . & . & . & . & . & .. \end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix} 0 & & & & & \\ & 0 & & & 0 & \\ & & 0 & & & \\ & & & m_1^2 & & \\ & 0 & & & m_2^2 & \\ & & & & & .. \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} m_1^2 & & & & \\ & m_2^2 & & 0 & \\ & & m_3^2 & & \\ & 0 & & .. & \\ & & & & .. \end{pmatrix} \tag{7}$$

the fermion ψ is written as

$$\psi_L = (1-\gamma_5)/2 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ . \end{pmatrix}, \quad \psi_R = (1+\gamma_5)/2 \begin{pmatrix} \psi_4 \\ \psi_5 \\ \psi_6 \\ . \\ . \end{pmatrix} \tag{8}$$

We thus have 3 massless left-handed $SU(2)$ doublets ψ_1, ψ_2, ψ_3, and an infinite series of vector-like massive $SU(2)$ doublets $\psi_4, \psi_5, \ldots$ with masses $m_1, m_2, ..$ as is seen in[1]

$$
\begin{aligned}
\mathcal{L}_L &= \bar{\psi}_1 i \not{D}(\frac{1-\gamma_5}{2})\psi_1 + \bar{\psi}_2 i \not{D}(\frac{1-\gamma_5}{2})\psi_2 \\
&\quad + \bar{\psi}_3 i \not{D}(\frac{1-\gamma_5}{2})\psi_3 \\
&\quad + \bar{\psi}_4(i \not{D} - m_1)\psi_4 + \bar{\psi}_5(i \not{D} - m_2)\psi_5 + \ldots
\end{aligned} \tag{9}
$$

An infinite number of right-handed fermions in a doublet notation are also introduced by(again in an abbreviated notation)

$$
\mathcal{L}_R = \overline{\phi} i\gamma^\mu(\partial_\mu - i(1/2)g' Y_R B_\mu)\phi - \overline{\phi}_L M' \phi_R - \overline{\phi}_R (M')^\dagger \phi_L \tag{10}
$$

where

$$
Y_R = \begin{pmatrix} 4/3 & 0 \\ 0 & -2/3 \end{pmatrix} \tag{11}
$$

for quarks and

$$
Y_R = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \tag{12}
$$

for leptons, and the mass matrix M' satisfies the index condition (6) but in general it may have different mass eigenvalues from those in(7). After the diagonalization of M' with non-zero eigenvalues $m'_1, m'_2, \ldots$, ϕ is written as

$$
\phi_L = (1-\gamma_5)/2 \begin{pmatrix} \phi_4 \\ \phi_5 \\ \phi_6 \\ \cdot \\ \cdot \end{pmatrix}, \quad \phi_R = (1+\gamma_5)/2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \cdot \end{pmatrix} \tag{13}
$$

[1] One may introduce <u>constant</u> complete orthonormal sets $\{u_n\}$ and $\{v_n\}$ defined by
$M^\dagger M u_n = 0$ for $n = -2, -1, 0,$
$M^\dagger M u_n = m_n^2, M M^\dagger v_n = m_n^2 v_n$ for $n = 1, 2, \ldots$
by assuming the index condition (6). One then has $M u_n = m_n v_n$ for $m_n \neq 0$ by choosing the phase of v_n and $M u_n = 0$ for $m_n = 0$. When one expands
$\psi_L = \sum_{n=-2}^{\infty} \psi_{n+3}^L u_n, \psi_R = \sum_{n=1}^{\infty} \psi_{n+3}^R v_n$
one recovers the mass matrix (7) and the relation (9).

Here, ϕ_1, ϕ_2,and ϕ_3 are right-handed and massless, and $\phi_4, \phi_5, ...$ have masses m'_1, m'_2,..

$$\begin{aligned}\mathcal{L}_R &= \bar{\phi}_1 i \not{D}(\frac{1+\gamma_5}{2})\phi_1 + \bar{\phi}_2 i \not{D}(\frac{1+\gamma_5}{2})\phi_2 \\ &+ \bar{\phi}_3 i \not{D}(\frac{1+\gamma_5}{2})\phi_3 \\ &+ \bar{\phi}_4(i \not{D} - m'_1)\phi_4 + \bar{\phi}_5(i \not{D} - m'_2)\phi_5 + ... \end{aligned} \tag{14}$$

with

$$\not{D} = \gamma^\mu(\partial_\mu - i(1/2)g'Y_R B_\mu) \tag{15}$$

The present model is vector-like and manifestly anomaly-free before the breakdown of parity (6);after the breakdown of parity,the model still stays anomaly-free provided that both of M and M' satisfy the index condition (6). In this scheme, the anomaly is caused by the left-right asymmetry, in particular, in the sector of (infinitely) heavy fermions; in this sense, the parity breaking (6) may be termed "hard breaking". Unlike conventional vector-like models with a finite number of components[7], the present scheme avoids the appearance of a strongly interacting right-handed sector despite of the presence of heavy fermions.

The massless fermion sector in the above scheme reproduces the same set of fermions as in the standard model. However, heavier fermions have distinct features. For example, the heavier fermion doublets with the smallest masses are described by

$$\begin{aligned}\mathcal{L} &= \overline{\psi}_4 i\gamma^\mu(\partial_\mu - igW^a_\mu - i(1/2)g'Y_L B_\mu)\psi_4 - m_1\overline{\psi}_4\psi_4 \\ &+ \overline{\phi}_4 i\gamma_\mu(\partial_\mu - i(1/2)g'Y_R B_\mu)\phi_4 - m'_1\overline{\phi}_4\phi_4 \end{aligned} \tag{16}$$

The spectrum of fermions is thus *doubled* to be vector-like in the sector of heavy fermions and ,at the same time, the masses of ψ and ϕ become non-degenerate, i.e., $m_1 \neq m'_1$. As a result, the fermion number anomaly[8] is generated only by the first 3 generations of light fermions;the violation of baryon number is not enhanced by the presence of heavier fermions. The masses of heavy doublet components in ψ are degenerate in the present zeroth order approximation. If one lets all the masses $m_1, m_2, ..., m'_1, m'_2, ...$ to ∞ in the above model, one recovers the standard model.

Apparently, the present mechanism of generating chiral fermions does not explain a basic dynamics which is responsible for the chiral structure. Nevertheless, this kinematical picture is attractive and might pave a way to a more fundamental understanding of the chiral structure.

If one assumes that those masses appearing in (9) and (14) are large but finite, for example, about a few TeV and heavier, one obtains a generalization of the conventional vector-like model. The creation of realistic non-vanishing masses for known light quarks and leptons , which are massless in the above scheme, by the Higgs mechanism and the physical implications of the model are discussed in Ref.[9].

III. Generalized Pauli-Villars regularization

The most important feature of the vector-like scheme described in Section 2 is that all the heavier fremions *decouple* in the limit of large fermion masses

$$\begin{aligned} m_1, m_2, \ldots &\rightarrow \infty \\ m' \ \cdots ' &\rightarrow \infty \end{aligned} \tag{17}$$

in (9) and (14). In the phenomenological level, this property ensures that those heavier fermions do not spoil the successful aspects of the Weinberg-Salam theory.

This decoupling of heavy fermions also implies that those fermions, if suitably formulated, can be used as regulator fields. In fact, the recent formulation of the generalized Pauli-Villars regularization of chiral gauge theory by Frolov and Slavnov [10] is based on this property, which in turn led to the vector-like formulation of Narayanan and Neuberger [6]. To be definite , the chiral theory which we want to regularize is defined by

$$= \overline{\psi} i \not{D} \left(1 + \gamma_5 \right) \psi \tag{18}$$

where

$$\begin{aligned} \not{D} &= \gamma^\mu(\partial_\mu - igA^a_\mu(x)T^a) \\ &\equiv \gamma^\mu(\partial_\mu - igA_\mu(x)) \end{aligned} \tag{19}$$

In the Euclidean metric we use , the Dirac operator $\not{D}$ is formally hermitian.

The generalized Pauli-Villars regularization of (18) is defined by

$$\begin{aligned}\mathcal{L} &= \overline{\psi} i \not{D} \psi - \overline{\psi}_L M \psi_R - \overline{\psi}_R M^\dagger \psi_L \\ &\quad + \overline{\phi} i \not{D} \phi - \overline{\phi} M' \phi \end{aligned} \tag{20}$$

where

$$\psi_R = \frac{1}{2}(1+\gamma_5)\psi \quad , \quad \psi_L = \frac{1}{2}(1-\gamma_5)\psi \tag{21}$$

and the infinite dimensional mass matrices in (20) are defined by

$$\begin{aligned}
M &= \begin{pmatrix} 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 6 & \cdots \\ \cdots & & & & \end{pmatrix} \Lambda \\
M^\dagger M &= \begin{pmatrix} 0 & & & & \\ & 2^2 & & 0 & \\ & & 4^2 & & \\ & 0 & & 6^2 & \\ & & & & \ddots \end{pmatrix} \Lambda^2 \\
M M^\dagger &= \begin{pmatrix} 2^2 & & & \\ & 4^2 & & 0 \\ & & 6^2 & \\ & 0 & & \ddots \end{pmatrix} \Lambda^2 \\
M' &= \begin{pmatrix} 1 & & & \\ & 3 & & 0 \\ & & 5 & \\ & 0 & & \ddots \end{pmatrix} \Lambda = (M')^\dagger
\end{aligned} \tag{22}$$

where Λ is a parameter with dimensions of mass. The mass matrix thus carries a unit index

$$dim\ ker\ M^\dagger M - dim\ ker\ M M^\dagger = 1 \tag{23}$$

The fields ψ and ϕ in (20) then contain an infinite number of components , each of which is a conventional 4-component Dirac field;$\psi(x)$ consists of conventional anti-commuting (Grassmann) fields , and $\phi(x)$ consists of commuting bosonic Dirac fields.

The Lagrangian (20) is invariant under the gauge transformation

$$\begin{aligned}
\psi(x) &\rightarrow \psi'(x) = U(x)\psi(x) \equiv exp[iw^a(x)T^a]\psi(x) \\
\overline{\psi}(x) &\rightarrow \overline{\psi}'(x) = \overline{\psi}(x)U(x)^{\dagger} \\
\phi(x) &\rightarrow \phi'(x) = U(x)\phi(x) \\
\overline{\phi}(x) &\rightarrow \overline{\phi}'(x) = \overline{\phi}(x)U(x)^{\dagger} \\
\not{D} &\rightarrow \not{D}' = U(x)\not{D}U(x)^{\dagger}.
\end{aligned} \tag{24}$$

The Noether current associated with the gauge coupling in (20) is defined by the infinitesimal change of matter variables in (24) with $\not{D}$ kept fixed :

$$\begin{aligned}
\mathcal{L}' &= \overline{\psi}'i\not{D}\psi' - \overline{\psi}'_L M\psi'_R - \overline{\psi}'_R M^{\dagger}\psi'_L \\
&\qquad + \overline{\phi}'i\not{D}\phi' - \overline{\phi}'M'\phi' \\
&= -(D_\mu w)^a J^{\mu a}(x) + \mathcal{L}
\end{aligned} \tag{25}$$

with

$$J^{\mu a}(x) = \overline{\psi}(x)T^a\gamma^\mu\psi(x) + \overline{\phi}(x)T^a\gamma^\mu\phi(x). \tag{26}$$

Similarly , the U(1) transformation

$$\begin{aligned}
\psi(x) &\rightarrow e^{i\alpha(x)}\psi(x)\ ,\ \overline{\psi}(x) \rightarrow \overline{\psi}(x)e^{-i\alpha(x)} \\
\phi(x) &\rightarrow e^{i\alpha(x)}\phi(x)\ ,\ \overline{\phi}(x) \rightarrow \overline{\phi}(x)e^{-i\alpha(x)}
\end{aligned} \tag{27}$$

gives rise to the U(1) fermion number current

$$J^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) + \overline{\phi}(x)\gamma^\mu\phi(x). \tag{28}$$

The chiral transformation

$$\begin{aligned}
\psi(x) &\rightarrow e^{i\alpha(x)\gamma_5}\psi(x)\ ,\ \overline{\psi} \rightarrow \overline{\psi}(x)e^{i\alpha(x)\gamma_5} \\
\phi(x) &\rightarrow e^{i\alpha(x)\gamma_5}\phi(x)\ ,\ \overline{\phi} \rightarrow \overline{\phi}(x)e^{i\alpha(x)\gamma_5}
\end{aligned} \tag{29}$$

gives the U(1) chiral current

$$J_5^\mu(x) = \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x) + \overline{\phi}(x)\gamma^\mu\gamma_5\phi(x). \tag{30}$$

Considering the variation of action under the transformation (25) and (27) , one can show that the vector currents (26) and (28) are <u>*naively*</u> conserved [2]

$$\begin{aligned} (D_\mu J^\mu)^a(x) &\equiv \partial_\mu J^{\mu a}(x) + g f^{abc} A_\mu^b(x) J^{\mu c}(x) = 0, \\ \partial_\mu J^\mu(x) &= 0 \end{aligned} \tag{31}$$

whereas the chiral current (30) satisfies the <u>*naive*</u> identity

$$\partial_\mu J_5^\mu(x) = 2i\overline{\psi}_L M\psi_R - 2i\overline{\psi}_R M^\dagger\psi_L + 2i\overline{\phi}M'\gamma_5\phi. \tag{32}$$

The quantum theory of (20) may be defined by the path integral , for example ,

$$< \overline{\psi}(x)T^a\gamma^\mu\psi(x) > = \int d\mu\overline{\psi}(x)T^a\gamma^\mu\psi(x)exp[\int \mathcal{L}d^4x]. \tag{33}$$

The path integral over the bosonic variables ϕ and $\overline{\phi}$ for the Dirac operator in Euclidean theory needs to be defined via a suitable rotation in the functional space.

Definition of currents in terms of propagators

We now define the currents in terms of propagators to clarify the basic mechanism of generalized Pauli-Villars regularization [11]. The basic idea of this approach is explained for the <u>*un − regularized*</u> theory in (18) as follows : We start with the current associated with the gauge coupling

[2]The fact that the regularized currents satisfy anomaly-free relations (31) shows that the regularization (20) is ineffective for the evaluation of possible anomalies in these vector currents. In particular, this scheme works only for anomaly-free gauge theory.

$$
\begin{aligned}
&< \overline{\psi}(x) T^a \gamma^\mu (\frac{1+\gamma_5}{2}) \psi(x) > \\
&= \lim_{y \to x} < T^* \overline{\psi}(y) T^a \gamma^\mu (\frac{1+\gamma_5}{2}) \psi(x) > \\
&= - \lim_{y \to x} < T^* (T^a)_{bc} \gamma^\mu_{\alpha\delta} (\frac{1+\gamma_5}{2})_{\delta\beta} \psi_{\beta c}(x) \overline{\psi}_{\alpha b}(y) > \\
&= \lim_{y \to x} Tr[T^a \gamma^\mu (\frac{1+\gamma_5}{2}) \frac{1}{i \not{D}} \delta(x-y)] \qquad (34)
\end{aligned}
$$

where we used the anti-commuting property of ψ and the expression of the propagator

$$
< T^* \psi(x) \overline{\psi}(y) > = (\frac{1+\gamma_5}{2}) \frac{(-1)}{i \not{D}_x} \delta(x-y) \qquad (35)
$$

The trace in (34) runs over the Dirac and Yang-Mills indices. We now notice the expansion

$$
\begin{aligned}
\frac{1}{i \not{D}} &= \frac{1}{i \not{\partial} + g \not{A}} \\
&= \frac{1}{i \not{\partial}} + \frac{1}{i \not{\partial}} (-g \not{A}) \frac{1}{i \not{\partial}} \\
&\quad + \frac{1}{i \not{\partial}} (-g \not{A}) \frac{1}{i \not{\partial}} (-g \not{A}) \frac{1}{i \not{\partial}} + \cdots \qquad (36)
\end{aligned}
$$

When one inserts (36) into (34) and retains only the terms linear in $A^b_\nu(x)$, one obtains

$$
\begin{aligned}
&\lim_{y \to x} Tr[T^a \gamma^\mu (\frac{1+\gamma_5}{2}) \frac{(-1)}{i \not{\partial}} \gamma^\nu T^b g A^b_\nu(x) \frac{1}{i \not{\partial}} \delta(x-y)] \\
&= \lim_{y \to x} \int d^4 z Tr[T^a \gamma^\mu (\frac{1+\gamma_5}{2}) \frac{(-1)}{i \not{\partial}} \\
&\qquad \times \delta(x-z) T^b \gamma^\nu \frac{1}{i \not{\partial}} \delta(x-y)] g A^b_\nu(z) \qquad (37)
\end{aligned}
$$

where the derivative ∂_μ acts on *all* the x- variables standing on the right of it in (37). If one takes the variational derivative of (37) with respect to $gA^b_\nu(z)$, one obtains

$$
\begin{aligned}
&\lim_{y\to x} Tr[T^a\gamma^\mu(\frac{1+\gamma_5}{2})\frac{(-1)}{i\not\partial}\delta(x-z)\gamma^\nu T^b\frac{1}{i\not\partial}\delta(x-y)] \\
&= \lim_{y\to x}\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}Tr[T^a\gamma^\mu(\frac{1+\gamma_5}{2})\frac{(-1)}{\not k+\not q}T^b\gamma^\nu\frac{1}{\not k}]e^{-iq(x-z)}e^{-ik(x-y)} \\
&= \int\frac{d^4q}{(2\pi)^4}e^{-iq(x-z)}(-1)\int\frac{d^4k}{(2\pi)^4}Tr[T^a\gamma^\mu(\frac{1+\gamma_5}{2})\frac{1}{\not k+\not q}T^b\gamma^\nu(\frac{1+\gamma_5}{2})\frac{1}{\not k}] \\
&\equiv \int\frac{d^4q}{(2\pi)^4}e^{-iq(x-z)}\Pi^{ab}_{\mu\nu}(q) \qquad (38)
\end{aligned}
$$

where we used the representations of δ-function

$$
\begin{aligned}
\delta(x-z) &= \int\frac{d^4q}{(2\pi)^4}e^{-iq(x-z)} \\
\delta(x-y) &= \int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}. \qquad (39)
\end{aligned}
$$

The last expression in (38) stands for the vacuum polarization tensor. Namely , one can generate the multiple correlation functions of currents $\overline{\psi}T^a\gamma^\mu(\frac{1+\gamma_5}{2})\psi$ in the perturbative sense by taking the variational derivative of (34) with respect to gauge fields A^a_μ . This idea also works for the non-gauge currents (28) and (30). We emphasize that we always take the limit $y = x$ first before the explicit calculation , and thus (34) differs from the point-splitting definition of currents.

We now generalize the above definition of currents for the theory defined by (20). For this purpose , we rewrite (20) as

$$\mathcal{L} = \overline{\psi}i\mathcal{D}\psi + \overline{\phi}i\mathcal{D}'\phi \qquad (40)$$

with

$$
\begin{aligned}
\mathcal{D} &\equiv \not D + iM(\frac{1+\gamma_5}{2}) + iM^\dagger(\frac{1-\gamma_5}{2}) \\
\mathcal{D}' &\equiv \not D + iM'. \qquad (41)
\end{aligned}
$$

The gauge current (26) is then defined by

$$\begin{aligned}
J^{\mu a}(x) &= \lim_{y\to x}\{< T^{*}\overline{\psi}(y)T^{a}\gamma^{\mu}\psi(x) > + < T^{*}\overline{\phi}(y)T^{a}\gamma^{\mu}\phi(x) >\} \\
&= \lim_{y\to x}\{- < T^{*}T^{a}\gamma^{\mu}\psi(x)\overline{\psi}(y) > + < T^{*}T^{a}\gamma^{\mu}\phi(x)\overline{\phi}(y) >\} \\
&= \lim_{y\to x} Tr[T^{a}\gamma^{\mu}(\frac{1}{i\mathcal{D}} - \frac{1}{i\mathcal{D}'})\delta(x-y)] \qquad (42)
\end{aligned}$$

where trace includes the sum over the infinite number of field components in addition to Dirac and Yang-Mills indices. The anti-commuting property of $\psi(x)$ and the commuting property of $\phi(x)$ are used in (42).

We next notice the relations

$$\begin{aligned}
\frac{1}{\mathcal{D}} &= \frac{1}{\mathcal{D}^{\dagger}\mathcal{D}}\mathcal{D}^{\dagger} \\
&= \frac{1}{\not{D}^{2} + \frac{1}{2}M^{\dagger}M(1+\gamma_{5}) + \frac{1}{2}MM^{\dagger}(1-\gamma_{5})}\mathcal{D}^{\dagger} \\
&= [(\frac{1+\gamma_{5}}{2})\frac{1}{\not{D}^{2}+M^{\dagger}M} + (\frac{1-\gamma_{5}}{2})\frac{1}{\not{D}^{2}+MM^{\dagger}}] \\
&\qquad \times[\not{D} - iM^{\dagger}(\frac{1+\gamma_{5}}{2}) - iM(\frac{1-\gamma_{5}}{2})] \\
\frac{1}{\mathcal{D}'} &= \frac{1}{(\mathcal{D}')^{\dagger}\mathcal{D}'}(\mathcal{D}')^{\dagger} \\
&= \frac{1}{\not{D}^{2} + (M')^{2}}(\not{D} - iM'). \qquad (43)
\end{aligned}$$

We thus rewrite (42) as

$$\begin{aligned}
&Tr\left[-iT^{a}\gamma^{\mu}(\frac{1}{\mathcal{D}} - \frac{1}{\mathcal{D}'})\delta(x-y)\right] \\
&= Tr\left\{-iT^{a}\gamma^{\mu}\left[(\frac{1+\gamma_{5}}{2})\sum_{n=0}^{\infty}\frac{1}{\not{D}^{2}+(2n\Lambda)^{2}}\right.\right. \\
&\qquad +(\frac{1-\gamma_{5}}{2})\sum_{n=1}^{\infty}\frac{1}{\not{D}^{2}+(2n\Lambda)^{2}} \\
&\qquad \left.\left. -\sum_{n=0}^{\infty}\frac{1}{\not{D}^{2}+[(2n+1)\Lambda]^{2}}\right]\not{D}\delta(x-y)\right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}Tr\left[-iT^a\gamma^\mu \sum_{n=-\infty}^{\infty}\frac{(-1)^n \not{D}^2}{\not{D}^2+(n\Lambda)^2}\frac{1}{\not{D}}\delta(x-y)\right] \\
&\qquad +\frac{1}{2}Tr\left[-iT^a\gamma^\mu\gamma_5\frac{1}{\not{D}}\delta(x-y)\right] \\
&= \frac{1}{2}Tr\left[T^a\gamma^\mu f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y)\right] \\
&\qquad +\frac{1}{2}Tr\left[T^a\gamma^\mu\gamma_5\frac{1}{i\not{D}}\delta(x-y)\right]
\end{aligned} \tag{44}$$

where we explicitly evaluated the trace over the infinite number of components and used the fact that the trace over an odd number of γ-matrices vanishes. We also defined $f(x^2)$ by

$$\begin{aligned}
f(x^2) &\equiv \sum_{n=-\infty}^{\infty}\frac{(-1)^n x^2}{x^2+(n\Lambda)^2} \\
&\quad \frac{(\pi x/\Lambda)}{sinh(\pi x/\Lambda)}.
\end{aligned} \tag{45}$$

This last expression of (45) as a sum of infinite number of terms is given in Ref.[10]. The regulator $f(x^2)$, which rapidly approaches 0 at $x^2=\infty$, satisfies

$$\begin{aligned}
f(0) &= 1 \\
x^2 f'(x^2) &= 0 \; for \; x\to 0 \\
f(+\infty) &= f'(+\infty)=f''(+\infty)=\cdots=0 \\
x^2 f'(x^2) & \quad 0 \;\; for \;\; x\to\infty.
\end{aligned} \tag{46}$$

The essence of the generalized Pauli-Villars regularization (20) is thus summarized in terms of regularized currents as follows:

$$\begin{aligned}
&< \bar{\psi}(x)T^a\gamma^\mu(\frac{1+\gamma_5}{2})\psi(x) >_{PV} \\
&= \lim_{y\to x}\left\{\frac{1}{2}Tr\left[T^a\gamma^\mu f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y)\right.\right.
\end{aligned}$$

$$+\frac{1}{2}Tr\left[T^a\gamma^\mu\gamma_5\frac{1}{i\not{D}}\delta(x-y)\right]\Big\} \tag{47}$$

$$< \overline{\psi}(x)\gamma^\mu(\frac{1+\gamma_5}{2})\psi(x) >_{PV}$$
$$= \lim_{y\to x}\Big\{\frac{1}{2}Tr\left[\gamma^\mu f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y)\right]$$
$$+\frac{1}{2}Tr\left[\gamma^\mu\gamma_5\frac{1}{i\not{D}}\delta(x-y)\right]\Big\} \tag{48}$$

$$< \overline{\psi}(x)\gamma^\mu\gamma_5(\frac{1+\gamma_5}{2})\psi(x) >_{PV}$$
$$= \lim_{y\to x}\Big\{\frac{1}{2}Tr\left[\gamma^\mu\gamma_5 f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y)\right]$$
$$+\frac{1}{2}Tr\left[\gamma^\mu\frac{1}{i\not{D}}\delta(x-y)\right]\Big\} \tag{49}$$

In the left-hand sides of (47)∼(49), the currents are defined in terms of the original fields appearing in (18). The vector $U(1)$ and axial-vector currents written in terms of the original fields in (18) are identical , but the regularized versions ,i.e., (48) and (49) are different. In particular , the vector $U(1)$ current,i.e.,(48) is not completely regularized. See also Refs.[6] and [12]. This reflects the different form of naive identities in (31) and (32) ; if all the currents are well regularized , the naive form of identities would also coincide. We emphasize that all the one-loop diagrams are generated from the (partially) regularized currents in (47) ∼ (49) ; in other words ,(47) ∼ (49) retain all the information of the generalized Pauli-Villars regularization (20).

IV. Generalized Pauli-Villars regularization and anomalies

As is seen in (47), the possible anomalous term of the gauge current which contains γ_5 is not regularized. The generalized Pauli-Villars regularization of chiral gauge theory thus works only for the theories which contain no gauge anomaly[10]. In an anomaly-free gauge theory such as the Weinberg-Salam theory, the U(1) fermion number anomaly is physical and interesting. In the generalized Pauli-Villars regularization in (20), the possible anomalous term of the U(1) current

$$< \overline{\psi}(x)\gamma^{\mu}(\frac{1+\gamma_5}{2})\psi(x) >_{PV} \tag{50}$$

is not regularized, since the term which contains γ_5 is not regularized in (48). On the other hand, the "axial-current"

$$< \overline{\psi}(x)\gamma^{\mu}\gamma_5(\frac{1+\gamma_5}{2})\psi(x) >_{PV} \tag{51}$$

which is identical to (50) in the un-regularized theory, is in fact different in the generalized Pauli-Villars regularization in (49) and the possible anomalous term containing γ_5 is regularized as

$$\begin{aligned} &< \overline{\psi}(x)\gamma^{\mu}\gamma_5(\frac{1+\gamma_5}{2})\psi(x) >_{PV} \\ &= \lim_{y\to x}\left\{\frac{1}{2}Tr\left[\gamma^{\mu}\gamma_5 f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y)\right]\right. \\ &\quad \left.+\frac{1}{2}Tr\ \gamma^{\mu}\frac{1}{i\not{D}}\delta(x-y)\right] \\ &\to \sum_n \phi_n(x)^{\dagger}\left[\gamma^{\mu}(\frac{\gamma_5}{2})f(\lambda_n^2/\Lambda^2)\frac{1}{i\lambda_n}\right]\phi_n(x) \end{aligned} \tag{52}$$

where we used the complete set defined by

$$\begin{aligned} \not{D}\phi_n(x) &\equiv \lambda_n\phi_n(x) \\ \int \phi_m(x)^{\dagger}\phi_n(x)d^4x &= \delta_{m,n} \\ \delta_{\alpha\beta}\delta(x-y) &\to \sum \phi_n(x)_{\alpha}\phi_n(y)^{\dagger}_{\beta} \end{aligned} \tag{53}$$

with α and β including Dirac and Yang-Mills indices. One can thus evaluate the fermion number anomaly by using the last expression of the "axial-current" (52) as

$$\begin{aligned} &\partial_{\mu} < \overline{\psi}(x)\gamma^{\mu}\gamma_5(\frac{1+\gamma_5}{2})\psi(x) >_{PV} \\ &= \sum_n\left[-(\not{D}\phi_n(x))^{\dagger}(\frac{\gamma_5}{2})f(\lambda_n^2/\Lambda^2)\frac{1}{i\lambda_n}\phi_n(x)\right. \end{aligned}$$

$$
\begin{aligned}
&+\phi_n(x)^\dagger(\frac{-\gamma_5}{2})f(\lambda_n^2/\Lambda^2)\frac{1}{i\lambda_n}(\not{D}\phi_n(x)) \\
&= i\sum_n \phi_n(x)^\dagger\gamma_5 f(\lambda_n^2/\Lambda^2)\phi_n(x) \\
&= iTr\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5 f(\not{D}^2/\Lambda^2)e^{ikx} \\
&= (\frac{ig^2}{32\pi^2})Tr\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad for \quad \Lambda\rightarrow\infty \qquad (54)
\end{aligned}
$$

where we used the relation $(\gamma_\mu)^\dagger = -\gamma_\mu$ in the present Euclidean metric. We also followed the calculational scheme in Ref.[13] in the last step of (54).

We thus recover the conventional covariant form of anomaly for the fermion number current [8]. This calculation of the fermion number anomaly in the generalized Pauli-Villars regularization was first performed by Aoki and Kikukawa [12] on the basis of Feynman diagrams.

V. Conclusion

The chiral structure is the most fundamental property of elementary fermions in modern unified gauge theory. The chiral anomaly, which is related to the chiral structure, is a subtle but profound phenomenon in field theory. It is interesting that the generalized Pauli-Villars regularization [10] successfully regularizes the Weinberg-Salam theory, although it requires an infinite number of regulator fields.

The notion of chiral anomaly is also closely related to the so-called U(1) problem and strong CP problem. In this connection, H. Banerjee and his collaborators continuously clarified the most fundamental aspects of the problem and suggested a careful reassessment of the path integration quantization of modern gauge theory itself [14].

On the occasion of the 60th birthday of Prof. H. Banerjee, I wish him many happy returns.

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